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Zero Location and nth Root Asymptotics of Sobolev Orthogonal Polynomials

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For a wide class of Sobolev orthogonal polynomials, it is proved that their zeros are contained in a compact subset of the complex plane and the asymptotic zero distribution is obtained. With this information, the nth root asymptotic behavior outside the compact set containing all the zeros is given. © 1999 Academic Press

1. INTRODUCTION

1. Let $\{\mu_k\}_{k=0}^m$ be a set of $m+1$ finite positive Borel measures. For each $k=0, ..., m$ the support Δ_k of μ_k is a compact subset of the real line $\mathbb{R}$. We will assume that Δ_0 contains infinitely many points. On the space of all polynomials, we consider

$$\langle p, q \rangle_S = \sum_{k=0}^{m} \int p^{(k)}(x)\, q^{(k)}(x)\, d\mu_k(x) = \sum_{k=0}^{m} \langle p^{(k)} q^{(k)} \rangle_{L_2(\mu_k)}, \qquad (1)$$

where p, q are polynomials. As usual, $f^{(k)}$ denotes the kth derivative of a function f. Obviously, (1) defines an inner product on the linear space of

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all polynomials. Therefore, a unique sequence of monic orthogonal polynomials is associated to it. By Q_n, we will denote the corresponding monic orthogonal polynomial of degree n. The sequence $\{Q_n\}$ is called the sequence of Sobolev monic orthogonal polynomials relative to (1).

Sobolev orthogonal polynomials have attracted much attention in the past two decades. Many papers on the subject deal with the algebraic aspect of the theory. Recently, some important results have been obtained regarding their asymptotic behavior. In this direction, we mention three papers of general character.

In [4], an important step was taken in the study of the so-called discrete Sobolev inner product; that is, when μ_0 is the only measure containing infinitely many points in its support. When $\mu_0' > 0$ a.e. on its support which consists of an interval, the authors find the relative asymptotic behavior between the Sobolev orthogonal polynomials and the orthogonal polynomials associated with μ_0 (in fact, they consider a more general class of product not necessarily positive definite). Thus, the asymptotic behavior of discrete Sobolev orthogonal polynomials is reduced to the case when the inner product solely contains the measure μ_0.

In [2], with $m=1$, the authors assume that $\mu_0, \mu_1 \in \mathbf{Reg}$ (in the sense defined in [6]) and that their supports are regular sets (a compact subset of the complex plane is said to be regular if the unbounded connected component of its complement is regular with respect to the Dirichlet problem). Under these assumptions, they find the asymptotic zero distribution of the zeros of the derivatives of the Sobolev orthogonal polynomials and also of the proper sequence of Sobolev orthogonal polynomials when $\Delta_0 \supset \Delta_1$.

Finally, in [5] with $m=1$, for a wide class of Sobolev products defined on smooth curves of the complex plane, the author gives the strong asymptotics of the corresponding Sobolev orthogonal polynomials.

In contrast with the case of classical orthogonality with respect to a measure, where it is easy to prove that the zeros of the orthogonal polynomials lie on the convex hull of the support of the measure, the location of the zeros of Sobolev orthogonal polynomials in the complex plane for general Sobolev inner products seems to be a difficult problem. Thus, it is not possible to derive from the results in [2], the (uniform) nth root asymptotic behavior of the Sobolev orthogonal polynomials.

The main question considered in this paper is the study of the location of the zeros of Sobolev orthogonal polynomials. Under general assumptions on the measures involved in the inner product, we prove that the zeros of the Sobolev orthogonal polynomials are contained in a compact subset of the complex plane. This is done in Section 2 making use of methods from the theory of bounded operators. In Section 3, following the ideas in [2], we extend some results of that paper to $m \geqslant 2$. This extension together with the results of Section 2 allow us to give the nth root asymptotic behavior of Sobolev orthogonal polynomials for a wide class of Sobolev orthogonal polynomials.

2. Before proceeding, let us fix some assumptions and additional notation. As above, (1) defines an inner product on the space P of all polynomials. The norm of $p \in \mathrm{P}$ is

$$\|p\|_S = \left(\sum_{k=0}^{m} \int (p^{(k)})^2 (x)\, d\mu_k(x) \right)^{1/2} = \left(\sum_{k=0}^{m} \|p^{(k)}\|^2_{L_2(\mu_k)} \right)^{1/2}. \tag{2}$$

We will denote by $(H_{2,m}, \|\cdot\|_S)$ the Banach space obtained completing the normed space $(\mathrm{P}, \|\cdot\|_S)$. As usual, this is done identifying all Cauchy sequences of polynomials whose difference tends to zero in the norm $\|\cdot\|_S$. Certainty, $H_{2,m}$ heavily depends on the measures involved in the inner product, but for simplicity in the notation we will not indicate it. For $f \in H_{2,m}$, $\|f\|_S$ is defined by continuity; that is,

$$\|f\|_S = \lim_{n \to \infty} \|p_n\|_S,$$

where $\{p_n\}$ is a representative of f. On $H_{2,m}$, we consider the inner product

$$\langle f, g \rangle_S = \tfrac{1}{2}[\, \|f+g\|_S^2 - \|f\|_S^2 - \|g\|_S^2], \qquad f, g \in H_{2,m}. \tag{3}$$

Therefore, $(H_{2,m}, \langle \cdot, \cdot \rangle_S)$ is a separable Hilbert space because by construction the space of polynomials is dense in it. In particular, we have the sequence $\{q_n\}$ of Sobolev orthonormal polynomials ($\langle q_n, q_k \rangle_S = \delta_{n,k}$) forms a complete basis in $(H_{2,m}, \langle \cdot, \cdot \rangle_S)$ and the Parseval identity takes place

$$\|f\|_S^2 = \sum_{k=0}^{\infty} \alpha_k^2, \qquad \alpha_k = \alpha_k(f) = \langle f, q_k \rangle_S, \qquad f \in H_{2,m}. \tag{4}$$

In virtue of the Riesz–Fischer Theorem, the application which places $f \in H_{2,m}$ in correspondence with $\{\alpha_n(f)\} \in \ell_2$ establishes an isometric isomorphism between $H_{2,m}$ and ℓ_2 (the space of all square summable sequences of real numbers).

Temporarily, we restrict our attention to sets of measures $\{\mu_k\}$, $k = 0, 1, ..., m$, with the property that $xf \in H_{2,m}$ for each $f \in H_{2,m}$. By $xf \in H_{2,m}$ we mean that if two Cauchy sequences of polynomials $\{p_n\}$ and $\{l_n\}$ are representatives of f (and, therefore, $\lim_{n \to \infty} \|p_n - l_n\|_S = 0$), then the sequences of polynomials $\{xp_n\}$ and $\{xl_n\}$ are also equivalent Cauchy sequences (in the sense that $\lim_{n \to \infty} \|xp_n - xl_n\|_S = 0$). The element in $H_{2,m}$ which they represent is what we denote xf. In this case, it is easy to verify that the application $Mf = xf$ from $H_{2,m}$ onto $H_{2,m}$ is linear.

This property is not always fulfilled. The first result below gives a class of inner products for which M is bounded. We say that the Sobolev inner product (1) is *sequentially dominated* if

$$\Delta_k \subset \Delta_{k-1}, \qquad k = 1, ..., m,$$

and

$$d\mu_k = f_{k-1}\, d\mu_{k-1}, \qquad f_{k-1} \in L_\infty(\mu_{k-1}), \qquad k = 1, ..., m.$$

Obviously, this is the case when all the measures in the inner product are equal.

THEOREM 1. *Assume that the Sobolev inner product* (1) *is sequentially dominated, then the application* $Mf = xf$ *defines a bounded linear operator on* $H_{2,m}$ *with norm*

$$\|M\| \leqslant (2[C_1^2 + (m+1)^2\, C_2])^{1/2}, \tag{5}$$

where

$$C_1 = \max_{x \in \Delta_0} |x|, \qquad C_2 = \max_{k=0, ..., m-1} \|f_k\|_{L_\infty(\mu_k)}.$$

The boundedness of the multiplication operator has an interesting consequence on the location of the zeros of Sobolev orthogonal polynomials.

THEOREM 2. *Assume that the application* $Mf = xf$ *defines a bounded linear operator from* $H_{2,m}$ *onto* $H_{2,m}$. *Then, all the zeros of the Sobolev orthogonal polynomials are contained in the disk* $\{z: |z| \leqslant 2\, \|M\|\}$.

We underline that in Theorem 2 the inner product does not have to be sequentially dominated. The boundedness of M is the only requirement. Therefore, it is of interest to find other (or less restrictive) sufficient conditions for the boundedness of this operator.

3. We mention some concepts needed to state the result on the asymptotic zero distribution of Sobolev orthogonal polynomials. For any polynomial q of exact degree n, we denote

$$\nu(q) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j},$$

where $z_1, ..., z_n$ are the zeros of q repeated according to their multiplicity, and δ_{z_j} is the Dirac measure with mass one at the point z_j. This is the so called normalized zero counting measure associated with q. In [6], the authors introduce a class **Reg** of regular measures. For measures supported on a compact set of the real line, they prove that (see Theorem 3.6.1) $\mu \in \mathbf{Reg}$ if and only if the orthogonal polynomials q_n (in the usual sense) with respect to μ have regular asymptotic zero distribution. That is, that in the weak star topology of measures

$$\lim_{n \to \infty} v(q_n) = \omega_\Delta,$$

where ω_Δ is the equilibrium measure of the support Δ of the measure μ. In case that Δ is regular, the measure μ belongs to **Reg** (see Theorem 3.2.3 in [6]) if and only if

$$\lim_{n \to \infty} \left(\frac{\|p_n\|_\Delta}{\|p_n\|_{L_2(\mu)}} \right)^{1/n} = 1 \tag{6}$$

for every sequence of polynomials $\{p_n\}$, deg $p_n \leqslant n$, $p_n \not\equiv 0$. Here and in the following $\|\cdot\|_\Delta$ denotes the supremum norm on Δ.

Given a compact set Δ of the complex plane, we denote by $C(\Delta)$ the logarithmic capacity of Δ and by $g_\Delta(z: \infty)$ the corresponding Green's function with singularity at infinity (see, e.g., [3, 6]).

In the following,

$$\Delta = \bigcup_{k=0}^{m} \Delta_k,$$

where Δ_k is the support of μ_k in (1). Assume that there exists $l \in \{0, ..., m\}$ such that $\bigcup_{k=0}^{l} \Delta_k = \Delta$, where Δ_k is regular, and $\mu_k \in \mathbf{Reg}$ for $k = 0, ..., l$. Under these assumptions, we say that the Sobolev inner product (1) is *l-regular*.

The next result is inspired in Theorem 1 and Corollary 3 of [2]. We are very grateful to A. B. J. Kuijlaars for providing us with an early draft of the paper as well as for useful discussions on the subject.

THEOREM 3. *Let the Sobolev inner product* (1) *be l-regular. Then for each fixed* $k = 0, ..., l$ *and for all* $j \geqslant k$

$$\overline{\lim_{n \to \infty}} \|Q_n^{(j)}\|_{\Delta_k}^{1/n} \leqslant C(\Delta). \tag{7}$$

For all $j \geqslant l$

$$\lim_{n \to \infty} \|Q_n^{(j)}\|_\Delta^{1/n} = C(\Delta) \tag{8}$$

and

$$\lim_{n\to\infty} \nu(Q_n^{(j)}) = \omega_\Delta, \tag{9}$$

in the weak star topology of measures.

If the inner product is sequentially dominated, then $\Delta_0 = \Delta$; therefore, if Δ_0 and μ_0 are regular the corresponding inner product is 0-regular. In the sequel, $\mathbb{Z}_+ = \{0, 1, ...\}$. An immediate consequence of Theorems 2 and 3 is the following.

THEOREM 4. *Assume that the Sobolev inner product is sequentially dominated and* 0*-regular. Then, for all* $j \in \mathbb{Z}_+$

$$\overline{\lim_{n\to\infty}} \, |Q_n^{(j)}(z)|^{1/n} = C(\Delta)\, e^{g_\Delta(z;\,\infty)} \tag{10}$$

for every $z \in \mathbb{C}$ *except for a set of capacity zero, and*

$$\lim_{n\to\infty} |Q_n^{(j)}(z)|^{1/n} = C(\Delta)\, e^{g_\Delta(z;\,\infty)}, \tag{11}$$

uniformly on each compact subset of $\mathbb{C}\setminus\{z\colon |z| \leqslant 2\,\|M\|\}$, *where* $\|M\|$ *satisfies* (5).

These results will be complemented in the sections below. In the rest of the paper, we maintain the notations and definitions introduced above.

2. ZERO LOCATION

We fix an inner product of the form (1). For simplicity in the notation, we write

$$\langle \cdot, \cdot \rangle_{L_2(\mu_k)} = \langle \cdot, \cdot \rangle_k, \qquad \|\cdot\|_{L_2(\mu_k)} = \|\cdot\|_k.$$

Proof of Theorem 1. First of all, we show that there exists a constant $C > 0$ such that for any polynomial p

$$\|xp\|_S \leqslant C\, \|p\|_S. \tag{12}$$

Take C_1 and C_2 as in the statement of this theorem. Straightforward calculations lead to the estimates

$$\begin{aligned}\|xp\|_S^2 &= \sum_{k=0}^{m} \|(xp)^{(k)}\|_k^2 = \sum_{k=0}^{m} \|xp^{(k)} + kp^{(k-1)}\|_k^2 \\ &\leqslant 2 \sum_{k=0}^{m} (\|xp^{(k)}\|_k^2 + k^2 \|p^{(k-1)}\|_k^2) \\ &\leqslant 2 \sum_{k=0}^{m} (C_1^2 \|p^{(k)}\|_k^2 + k^2 C_2 \|p^{(k-1)}\|_{k-1}^2) \\ &\leqslant 2[C_1^2 + (m+1)^2 C_2] \sum_{k=0}^{m} \|p^{(k)}\|_k^2 = C^2 \|p\|_S^2,\end{aligned}$$

which imply (12) with

$$C = (2[C_1^2 + (m+1)^2 C_2])^{1/2}.$$

Let $f \in H_{2,m}$ and assume that $\{p_n\}$ is a representative of f. Using (12), for all $n, m \in \mathbb{Z}_+$ we have

$$\|xp_n - xp_m\|_S \leqslant C \|p_n - p_m\|_S.$$

This shows that $\{xp_n\}$ is also a Cauchy sequence. Moreover, if $\{l_n\}$ also represents f, from (12) we also have that for all $n \in \mathbb{Z}_+$

$$\|xp_n - xl_n\|_S \leqslant C \|p_n - l_n\|_S,$$

which shows that both sequences $\{xp_n\}$ and $\{xl_n\}$ represent the same element in $H_{2,m}$. This element is what we defined as xf in Section 1.

If $\{p_n\}$ is a representative of $f \in H_{2,m}$ and $\{l_n\}$ is a representative of $g \in H_{2,m}$, and $\alpha, \beta \in \mathbb{R}$ it is easy to verify, that $\{\alpha xp_n + \beta xl_n\}$ represents $x(\alpha f + \beta g)$ which amounts to the linearity of M. The boundedness of the operator follows immediately because (12) and the definition of the $\|\cdot\|_S$ norm give

$$\|xf\|_S = \lim_{n\to\infty} \|xp_n\|_S \leqslant C \lim_{n\to\infty} \|p_n\|_S = C \|f\|_S.$$

With this we conclude the proof of Theorem 1. ∎

Our next goal is to connect the operator M with an infinite Hessenberg matrix. We have that $H_{2,m}$ is isometrically isomorphic to ℓ_2 through the application which identifies an element $f \in H_{2,m}$ with the sequence of its Fourier coefficients (see (4)). Thus the nth Sobolev orthonormal polynomial q_n is in correspondence with the element e_n of ℓ_2 with 1 at the coordinate $n+1$ and the rest of the coordinates equal to 0. Since the sequence $\{q_n\}$ of orthonormal polynomials with respect to the inner product

$\langle \cdot, \cdot \rangle_S$ forms a basis in the space of all polynomials, we have that for each $n \in \mathbb{Z}_+$

$$xq_{n-1}(x) = \sum_{k=0}^{n} c_{k,n-1} q_k(x), \tag{13}$$

where

$$c_{k,n-1} = \langle xq_{n-1}, q_k \rangle_S, \qquad k = 0, ..., n.$$

From (13) we obtain that the matrix representation of M, taking in ℓ_2 the canonical basis $\{e_n\}$, is given by the infinite Hessenberg matrix

$$\mathcal{M} = \begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} & \cdots & c_{0,n-2} & c_{0,n-1} & \cdots \\ c_{1,0} & c_{1,1} & c_{1,2} & \cdots & c_{1,n-2} & c_{1,n-1} & \cdots \\ 0 & c_{2,1} & c_{2,2} & \cdots & c_{2,n-2} & c_{2,n-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2,n-2} & c_{n-2,n-1} & \cdots \\ 0 & 0 & 0 & \cdots & c_{n-1,n-2} & c_{n-1,n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

By $\mathcal{M}_n$, we denote the nth principal section of $\mathcal{M}$, and

$$\bar{q}_n(x) = (q_0(x), q_1(x), ..., q_{n-1}(x))^t.$$

Here and in the following $(\cdot)^t$ denotes the transpose of the vector or matrix $(\cdot)$. Relation (13) for consecutive values of n indicates that

$$x\bar{q}_n(x) = \mathcal{M}_n^t \bar{q}_n(x) + c_{n,n-1}(0, ..., 0, q_n(x))^t. \tag{15}$$

LEMMA 1. *Each zero λ of q_n is an eigenvalue of $\mathcal{M}_n^t$ and $\bar{q}_n(\lambda)$ is an associated eigenvector.*

Proof. Let $\mathcal{I}_n$ denote the identity matrix of order n. Evaluating (15) at point λ, we obtain that

$$(\mathcal{M}_n^t - \lambda \mathcal{I}_n)\, \bar{q}_n(\lambda) = 0.$$

This proves the assertion of the lemma. ∎

LEMMA 2. *Assume that M defines a bounded linear operator on $H_{2,m}$. Then, the infinite Hessenberg matrix $\mathcal{M}$ defines a bounded linear operator on*

ℓ_2 and $\|\mathcal{M}\| = \|M\|$. Moreover, if $\mathcal{M}_{n,\infty}$ denotes the infinite matrix which is obtained adding zeros to $\mathcal{M}_n$, then for all $n \in \mathbb{Z}_+$

$$\|\mathcal{M}_{n,\infty}\| \leqslant 2\,\|M\|. \tag{16}$$

Proof. As pointed out above, $H_{2,m}$ and ℓ_2 are isometrically isomorphic, and $\mathcal{M}$ is the matrix representation of the operator M on the orthonormal basis of $H_{2,m}$ (see (13) and (14)). It immediately follows that $\|\mathcal{M}\| = \|M\|$.

In order to prove (16), notice that Schwarz's inequality and the boundedness of M give

$$|c_{n,n-1}| = |\langle xq_{n-1}, q_n\rangle_S| \leqslant \|xq_{n-1}\|_s \leqslant \|M\|.$$

For any $\bar{\alpha} \in \ell_2$, let $\bar{\alpha}_n$ denote its projection over the space generated by the first $n+1$ elements $e_0, ..., e_n$ of the canonical basis in ℓ_2. It is easy to verify that

$$\mathcal{M}_{n,\infty}\bar{\alpha}^t = \mathcal{M}_{n,\infty}\bar{\alpha}^t_{n-1} = \mathcal{M}\bar{\alpha}^t_{n-1} - c_{n,n-1}\alpha_{n-1}e^t_n.$$

Therefore,

$$\|\mathcal{M}_{n,\infty}\bar{\alpha}^t\|_{\ell_2} \leqslant \|\mathcal{M}\bar{\alpha}^t_{n-1}\|_{\ell_2} + |c_{n,n-1}\alpha_{n-1}| \leqslant 2\,\|M\|\;\|\bar{\alpha}\|_{\ell_2},$$

which gives (16). ∎

Proof of Theorem 2. According to Lemma 1, all the zeros of q_n are eigenvalues of $\mathcal{M}_n$. Obviously, the eigenvalues of $\mathcal{M}_n$ are eigenvalues of $\mathcal{M}_{n,\infty}$ and because of (16), for all $n \in \mathbb{Z}_+$, the spectrum of $\mathcal{M}_{n,\infty}$ is completely contained in the disk $\{z: |z| \leqslant 2\,\|M\|\}$. With this we conclude the proof of Theorem 1. ∎

A direct consequence of Theorems 1 and 2 is

COROLLARY 1. *Assume that the Sobolev inner product* (1) *is sequentially dominated, then all the zeros of the Sobolev orthogonal polynomials are contained in* $\{z: |z| \leqslant 2\,\|M\|\}$, *where* $\|M\|$ *satisfies* (5).

3. REGULAR ASYMPTOTIC ZERO DISTRIBUTION

For the proof of Theorem 3, we need the following lemma.

LEMMA 3. *Let E be a compact regular subset of the complex plane and $\{P_n\}$ a sequence of polynomials such that* $\deg P_n \leqslant n$ *and $P_n \not\equiv 0$. Then, for all $k \in \mathbb{Z}_+$,*

$$\overline{\lim_{n\to\infty}} \left(\frac{\|P_n^{(k)}\|_E}{\|P_n\|_E}\right)^{1/n} \leqslant 1. \tag{17}$$

Proof. Since P_n appears in the numerator and the denominator of the expression above, we can assume without loss of generality that P_n is monic. Fix an arbitrary $\varepsilon > 0$. Consider the curve $\gamma_\varepsilon = \{z \in \mathbb{C} : g_E(z; \infty) = \varepsilon\}$, where $g_E(z; \infty)$ denotes Green's function with respect to the unbounded connected component of the complement of E with singularity at infinity. The curve γ_ε is closed and analytic, thus it has finite length l_ε and it is at a distance $d > 0$ from E. Since E is regular, the curve γ_ε surrounds E. By Cauchy's integral formula and the Bernstein–Walsh Lemma, we have that for each $z \in E$

$$|P_n^{(k)}(z)| = \left| \frac{k!}{2\pi i} \int_{\gamma_\varepsilon} \frac{P_n(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta \right| \leqslant \frac{k!}{2\pi} \int_{\gamma_\varepsilon} \frac{|P_n(\zeta)|}{|\zeta - z|^{k+1}} \, |d\zeta|$$

$$\leqslant \frac{k! \, l_\varepsilon}{2\pi d^{k+1}} \, \|P_n\|_{\gamma_\varepsilon} \leqslant \frac{k! \, l_\varepsilon}{2\pi d^{k+1}} \, \|P_n\|_E \, e^{n\varepsilon}.$$

Therefore,

$$\left(\frac{\|P_n^{(k)}\|_E}{\|P_n\|_E} \right)^{1/n} \leqslant \left(\frac{k! \, l_\varepsilon}{2\pi d^{k+1}} \right)^{1/n} e^{\varepsilon},$$

and

$$\overline{\lim_{n \to \infty}} \left(\frac{\|P_n^{(k)}\|_E}{\|P_n\|_E} \right)^{1/n} \leqslant e^{\varepsilon}.$$

Making $\varepsilon \to 0$, (17) follows immediately. ∎

Proof of Theorem 3. We start out showing that

$$\overline{\lim_{n \to \infty}} \, \|Q_n\|_S^{1/n} \leqslant C(\Delta). \tag{18}$$

Since each of the sets Δ_k, $k = 0, ..., l$ is regular, so is Δ. Let T_n denote the monic Chebyshev polynomial of degree n for the set Δ. It is well known that $\lim_{n \to \infty} \|T_n\|_\Delta^{1/n} = C(\Delta)$. Then, by Lemma 2, for all $j \in \mathbb{Z}_-$

$$\overline{\lim_{n \to \infty}} \, \|T_n^{(j)}\|_\Delta^{1/n} \leqslant C(\Delta). \tag{19}$$

Therefore, by the minimizing property of the Sobolev norm of the polynomial Q_n, we have

$$\|Q_n\|_S^2 \leqslant \|T_n\|_S^2 = \sum_{k=0}^{m} \|T_n^{(k)}\|_k^2 \leqslant \sum_{k=0}^{m} \mu_k(\Delta_k) \, \|T_n^{(k)}\|_\Delta^2.$$

This estimate, together with (19), gives (18).

From the regularity of the measure μ_k (see (6)), we know that for each $k=0, ..., l$

$$\lim_{n\to\infty} \left(\frac{\| Q_n^{(k)} \|_{\Delta_k}}{\| Q_n^{(k)} \|_k} \right)^{1/n} = 1. \tag{20}$$

Since

$$\| Q_n^{(k)} \|_k \leqslant \| Q_n \|_S,$$

(18) and (20) imply

$$\overline{\lim_{n\to\infty}} \| Q_n^{(k)} \|_{\Delta_k}^{1/n} \leqslant C(\Delta). \tag{21}$$

Taking into consideration Lemma 3, relation (7) follows from (21).

If $j \geqslant l$, (7) takes place for each $k=0, ..., l$. Since

$$\| Q_n^{(j)} \|_{\Delta} = \max_{k=0, ..., l} \| Q_n^{(j)} \|_{\Delta_k},$$

using (7), we obtain

$$\overline{\lim_{n\to\infty}} \| Q_n^{(j)} \|_{\Delta}^{1/n} \leqslant C(\Delta).$$

But

$$\underline{\lim_{n\to\infty}} \| Q_n^{(j)} \|_{\Delta}^{1/n} \geqslant C(\Delta)$$

is always true for any sequence $\{Q_n\}$ of monic polynomials. Hence (8) follows.

The compact set Δ has empty interior and connected complement. It is well known (see, e.g., [1]) that under such conditions (8) implies (9). ∎

The so called discrete Sobolev orthogonal polynomials have attracted particular attention in the past years. They are of the form

$$\langle f, g \rangle_S = \int fg \, d\mu_0 + \sum_{i=1}^{m} \sum_{j=0}^{N_i} A_{i,j} f^{(j)}(c_i) \, g^{(j)}(c_i), \tag{22}$$

where $A_{i,j} \geqslant 0$, $A_{i,N_i} > 0$, and $c_i \in \mathbb{R}$. If any of the points c_i lie in the complement of the support Δ_0 of μ_0, the corresponding Sobolev inner product cannot be l-regular. Nevertheless, a simple modification of the proof of Theorem 3 allows to consider this case.

THEOREM 5. *Let the discrete Sobolev inner product* (22) *be such that* Δ_0 *is regular, and* $\mu_0 \in \mathbf{Reg}$. *Then*, (8)–(9) *take place, for all* $j \geqslant 0$, *with* $\Delta = \Delta_0$.

Proof. Let T_n denotes the nth monic Chebyshev polynomial with respect to Δ_0. Set

$$w(z) = \prod_{i=1}^{m} (z - c_i)^{N_i + 1}.$$

Let $N = \deg w$, and take $n \geqslant N$. Then,

$$\|Q_n\|_0^2 \leqslant \|Q_n\|_S^2 \leqslant \|wT_{n-N}\|_S^2 = \int |wT_{n-N}|^2 \, d\mu_0$$
$$\leqslant \mu_0(\Delta_0) \, \|w\|_{\Delta_0}^2 \, \|T_{n-N}\|_{\Delta_0}^2.$$

Since $\mu_0(\Delta_0)\|w\|_{\Delta_0}^2 > 0$ does not depend on n, we find that

$$\overline{\lim_{n \to \infty}} \, \|Q_n\|_0^{1/n} \leqslant C(\Delta_0).$$

From the regularity of the measure μ_0, it follows that

$$\overline{\lim_{n \to \infty}} \, \|Q_n\|_{\Delta_0}^{1/n} \leqslant C(\Delta_0).$$

Using the regularity of the compact set Δ_0 and Lemma 3 (for $E = \Delta_0$), we obtain that

$$\overline{\lim_{n \to \infty}} \, \|Q_n^{(j)}\|_{\Delta_0}^{1/n} \leqslant C(\Delta_0),$$

for all $j \geqslant 0$. This inequality is necessary and sufficient in order that (8) takes place (with $\Delta = \Delta_0$), which in turn implies (9). ∎

Proof of Theorem 4. From Corollary 1, we have that for all $n \in \mathbb{Z}_+$, the zeros of the Sobolev orthogonal polynomials are contained in a compact subset of the complex plane. It is well known that the zeros of the derivative of a polynomial lie in the convex hull of the set of zeros of the polynomial itself. Therefore, there exists a compact subset of the complex plane containing the zeros of $Q_n^{(j)}$ for all n, $j \in \mathbb{Z}_+$. In particular, all these zeros are contained in $\{z \colon |z| \leqslant 2 \, \|M|\}$. Thus, for each fixed $j \in \mathbb{Z}_+$ the measures $\nu_{n,j} = \nu(Q_n^{(j)})$, $n \in \mathbb{Z}_+$, and ω_Δ have their support contained in a compact subset of $\mathbb{C}$. Using this and (9) of Theorem 3, from the lower envelope theorem (see [6, p. 223]), we obtain

$$\underline{\lim}_{n \to \infty} \int \log \frac{1}{|z - x|} \, d\nu_{n,j}(x) = \int \log \frac{1}{|z - x|} \, d\omega_\Delta(x),$$

for all $z \in \mathbb{C}$ except for a set of zero capacity. This limit is equivalent to (10) because (see [6, p. 7])

$$g_{\Delta}(z\colon \infty) = \log \frac{1}{C(\Delta)} - \int \log \frac{1}{|z-x|}\, d\omega_{\Delta}(x).$$

In order to prove (11), notice that for each fixed $j \in \mathbb{Z}_+$, the family of functions

$$\left\{ \int \log \frac{1}{|z-x|}\, d\nu_{n,j}(x) \right\}, \qquad n \in \mathbb{Z}_+,$$

is formed by harmonic functions in the variable z which are uniformly bounded on each compact subset of $D = \mathbb{C} \setminus \{z\colon |z| \leqslant 2\,\|M\|\}$. From (10), we have that any subsequence which converges uniformly on compact subsets of D must tend to $\int \log |z-x|^{-1}\, d\omega_{\Delta}(x)$ (independent of the convergent subsequence which was chosen). Therefore, the whole sequence converges uniformly on compact subsets of D to this function. This is equivalent to (11). ∎

To conclude, we give another consequence of Theorem 3 and Corollary 1.

THEOREM 6. *Assume that the Sobolev inner product is sequentially dominated and* 0-*regular. Then, for all* $j \in \mathbb{Z}_-$

$$\lim_{n\to\infty} \frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} = \int \frac{d\omega_{\Delta}(x)}{z-x}, \tag{23}$$

uniformly on compact subsets of $\mathbb{C} \setminus \{z\colon |z| \leqslant 2\,\|M\|\}$, *where* $\|M\|$ *satisfies* (5).

Proof. Let $x_{n,i}^{j}$, $i = 1, ..., n-j$, denote the $n-j$ zeros of $Q_n^{(j)}$. As mentioned above, all these zeros are contained in $\{z\colon |z| \leqslant 2\,\|M\|\}$. Decomposing in simple fractions and using the definition of $\nu_{n,j}$, we obtain

$$\frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} = \frac{1}{n} \sum_{i=1}^{n-j} \frac{1}{z - x_{n,i}^{j}} = \frac{n-j}{n} \int \frac{d\nu_{n,j}(x)}{z-x}. \tag{24}$$

Therefore, for each fixed $j \in \mathbb{Z}_+$, the family of functions

$$\left\{ \frac{Q_n^{(j+1)}(z)}{nQ_n^{(j)}(z)} \right\}, \qquad n \in \mathbb{Z}_+, \tag{25}$$

is uniformly bounded on each compact subset of $D = \mathbb{C} \setminus \{z\colon |z| \leqslant 2\,\|M\|\}$.

On the other hand, all the measure $\nu_{n,j}$, $n \in \mathbb{Z}_+$, are supported in $\{z: |z| \leqslant 2\, \|M\|\}$ and for $z \in D$ fixed, the function $(z-x)^{-1}$ is continuous on $\{z: |z| \leqslant 2\, \|M\|\}$ with respect to x. Therefore, from (9) and (24), we find that any subsequence of (25) which converges uniformly on compact subsets of D converges pointwise to $\int (z-x)^{-1}\, d\omega_\Delta(x)$. Thus, the whole sequence converges uniformly on compact subsets of D to this function as stated in (23). ∎

Due to Theorem 5, results analogous to Theorems 4 and 6 may be obtained for discrete Sobolev orthogonal polynomials. For this, we must add to the restrictions of Theorem 5 that in (22) all $A_{i,j}$ be greater than zero in order that the corresponding inner product be sequentially dominated. Nevertheless, in any discrete Sobolev inner product, it is easy to see that at least $n-N-j$ zeros of $Q_n^{(j)}$ lie in the open convex hull of Δ_0 (where N is as in the proof of Theorem 5). Thus, if one eliminates from $Q_n^{(j)}$ the factors corresponding to zeros lying outside the convex hull of Δ_0 the remaining polynomial satisfies formulas similar to (11) and (23) (even if some $A_{i,j}=0$). We leave to the reader the statement of the corresponding results.

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